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Primitive Sets of F_n / R Lie Algebras

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Abstract: Let $\boldsymbol{F_n}$ be the free Lie algebra freely generated by a set $\{\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_n}\}$ and let $\boldsymbol{R}$ be a verbal ideal of $\boldsymbol{F_n}$. We prove that if $\boldsymbol{W}$ is a primitive subset of $\boldsymbol{F_n/R}$ which all of its elements do not involve $\boldsymbol{x_n}$ then $\boldsymbol{W}$ is primitive in $\boldsymbol{F_{n-1}/\widehat{R}}$, where $\boldsymbol{\widehat{R} = R \cap F_{n-1}}$.

Keywords: Free Lie algebras, solvable, nilpotent (super)algebras.

1. INTRODUCTION

Let F_n be a free Lie algebra over the field K of characteristic zero with a finite free generating set $X = \{x_1, x_2, \cdots, x_n\}$. Derived terms of the derived series of F_n, lower central series of F_n and more generally verbal ideals of F_n are fully invariant ideals. If R be a verbal ideal of F_n then the quotients of the form F_n/R include the free Lie algebra, free nilpotent Lie algebra and free solvable Lie algebra.

Let $\{a_1, a_2, \cdots, a_k\}$, $k \leq n$, be a subset of F_n. The subset $\{a_i + R : i = 1,2, \cdots, k\}$ is primitive in F_n/R if it can be extended to a free generating set of F_n/R.

In this work we prove that a subset $\{a_1, a_2, \cdots, a_k\}$ of F_{n-1} if $\{a_1 + R,\ a_2 + R,\ \cdots, a_k + R\}$ is primitive in F_n/R then $\{a_1 + \hat{R},\ a_2 + \hat{R},\ \cdots, a_k + \hat{R}\}$ is primitive in $F_{n-1}/\hat{R}$, where F_{n-1} is the free Lie algebra generated by the set $\{x_1, x_2, \cdots, x_{n-1}\}$ and $\hat{R} = R \cap F_{n-1}$. We obtain our main results for different choices of R.

The motivation for this work came from the analogous results for free groups [3,4,5,6]. All background and undefined notions here can be found [1] and [8].

By $\gamma_m(F_n)$ and $\delta^m(F_n)$, we denote the m −th term of the lower central series and m −th term of the derived series of F_n, respectively. For the second term $\delta^2(F_n)$, we use $F_n^{''}$.

We regard all algebras in this work as given over an arbitrary fixed field K of characteristic zero.

2. PRELIMINARIES

In the proof of our results we use a interpretation of primitivity using Fox derivatives via a theorem of Umirbaev.

By $U(F_n)$ we define the universal enveloping algebra of F_n, i.e., the free associative algebra over the field K with the same set X of free generators. To define non-commutative Jacobian matrix, we have to introduce non-commutative partial derivatives. We call them Fox derivatives in honour of R. Fox who considered them in a free group ring [2].

There is the augmentation homomorphism $\varepsilon : U(F_n) \to K$ defined by $\varepsilon(x_i) = 0,\ 1 \leq i \leq n$. The kernel Δ of this homomorphism is a free left $U(F_n)$ −module with a free basis X, so that every element $u \in \Delta$ can be uniquely written in the form $u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} x_i$. The coordinates $\frac{\partial u}{\partial x_i}$ of the element u in the free basis X are called Fox derivatives. One can extend these derivations to the whole $U(F_n)$ by defining $\frac{\partial (1)}{\partial x_i} = 0$.

Let R be an ideal of F_n. Then by Δ_R we denote the right ideal of $U(F_n)$ generated by R.

We need the following well known technical lemmas.

Lemma 2.1. Let J be an arbitrary ideal of $U(F_n)$ and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in J$ for each $i,\ 1 \leq i \leq n$.

Lemma 2.2. Let R be an ideal of $\boldsymbol{F_n}$ and let $u \in F_n$. Then $u \in \Delta_R\Delta$ if and only if $u \in R^{''}$.

Now for an arbitrary finite set of elements $Y = \{y_1, y_2, \cdots, y_k\} \subset U(F_n)$, we can define the matrix

$$J(Y) = \left(\frac{\partial y_i}{\partial x_j}\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}},$$

the Jacobian matrix. Essentially new properties of non-commutative Jacobian matrix have been discovered in [7], [11] and [13]. In [12], Umirbaev has proved a criterion of primitiveness for a system of elements in a finitely generated

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relatively free Lie algebra. About three years later in [9] it is shown that the subalgebra of a Lie algebra generated by a finite set of elements is equal to the maximal number of left independent rows of the corresponding Jacobian matrix.

These results reveal the following remarkable situation with the Jacobian matrix of a finite set of elements: "The invertibility of the Jacobian matrix means that a given set of elements is a part of a free generating set."

Theorem 2.3. [12] Let T be an arbitrary ideal of F_n. Then the following conditions are equivalent:

1. The system $E = \{f_1, f_2, \cdots, f_k\}$ of the algebra $F_n/\gamma_{c+1}(T)$ is primitive, where $c \geq 1$.
2. The Jacobian matrix $J(E)$ is right invertible over $U(F_n/T)$.
3. The minors of order k of the matrix $J(E)$ generate the unitary ideal of the algebra $U(F_n/T)$.

Now taking $T = \{0\}$ we obtain the following.

Corollary 2.4. The system $Y = \{u_1, u_2, \cdots, u_k\}$ of the algebra F_n is primitive if and only if the Jacobian matrix $J(Y)$ is right invertible over $U(F_n)$.

3. MAIN RESULTS

Through this work F_n and F_{n-1} denote the free Lie algebras with the generating sets $\{x_1, x_2, \cdots, x_n\}$ and $\{x_1, x_2, \cdots, x_{n-1}\}$ respectively. Let R be a verbal ideal of F_n. We will consider our problem for different choices of R.

3.1. The Free Lie Algebra Case: $R = \{0\}$

Although the general case of the following theorem is given [10] we give a different and elegant proof.

Theorem 3.1. Let $W = \{a_1, a_2, \cdots, a_k\} \subset F_{n-1}$ be primitive in F_n. Then W is primitive in F_{n-1}.

Proof. Since the set W is primitive in F_n by Corollary 2.4 the Jacobian matrix $J(W)$ is right invertible over $U(F_n)$, i.e., there exists a $n \times k$ matrix $B = (b_{ij})$ with $(b_{ij}) \in U(F_n)$ satisfying

$$J(W)B = I_k, \qquad (1)$$

where I_k is the $k \times k$ identity matrix over $U(F_n)$. Since $W \subset F_{n-1}$, the elements of W do not involve x_n. The n −th column of the matrix $J(W)$ do not involve x_n. Thus the form of $J(W)$ is

$$J(W) = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \cdots & \frac{\partial a_1}{\partial x_{n-1}} & 0 \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \cdots & \frac{\partial a_2}{\partial x_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial a_k}{\partial x_1} & \frac{\partial a_k}{\partial x_2} & \cdots & \frac{\partial a_k}{\partial x_{n-1}} & 0 \end{pmatrix}.$$

Each entry of the matrix Bcan be uniquely written in the form $b_{ij} = q_{ij} + p_{ij}$, where q_{ij} represents the sum of all terms in b_{ij} involving x_n and p_{ij} is an element of $U(F_n)$, not involving x_n. Assign $Q = (q_{ij})$ and $P = (p_{ij})$. Then we have $B = Q + P$. By equality (1)

$$J(W).Q + J(W).P = I_k.$$

Either each entry of $J(W).Q$ is zero or involve x_n. Since neither I_k nor $J(W).P$ involves x_n we must have $J(W).Q = 0$ yielding $J(W).P = I_k$. Let $\bar{J}(W)$ and $\bar{P}$ be the matrices obtained from $J(W)$ and P, respectively, by deleting n −th column and row. Then $\bar{J}(W)$ is the Jacobian matrix of W over $U(F_{n-1})$ and $\bar{P}$ is a matrix over $U(F_{n-1})$. The equation $J(W).P = I_k$ implies that $\bar{J}(W).\overline{P} = I_k$. This shows that $\bar{J}(W)$ is right invertible over $U(F_{n-1})$. By Corollary 2.4 W is primitive in F_{n-1}.

3.2. The Free Metabelian Case: $R = F_n''$

Let $M_n = F_n/F_n''$ and $M_{n-1} = F_{n-1}/F_{n-1}''$.

Theorem 3.2. Let $W = \{a_1 + F_{n-1}'', a_2 + F_{n-1}'', \cdots, a_k + F_{n-1}''\}$ be a subset of M_{n-1}. If t $\bar{W} = \{a_1 + F_n'', a_2 + Fn'',\cdots,ak+Fn''$ is primitive in Mn then W is primitive in $Mn-1.$

Proof. Let $\bar{W}$ be a primitive subset of M_n. From Theorem 2.3 $\bar{J}(W)$ is right invertible over $U(F_n/F_n'')$. We consider matrices B, Q, P, $\bar{J}(W)$. $\bar{P}$ as in the proof Theorem 3.1. Now we are going to prove that

$$\bar{J}(W).\bar{P} = I_k.$$

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It follows from Lemma 2.1 and Lemma 2.2 that

$$J(\overline{W}) = \left(\frac{\partial}{\partial x_j}(a_i + F_n^{''})\right)_{\substack{1\le i\le k\\1\le j\le n}} = \left(\frac{\partial a_i}{\partial x_j} + \Delta_{F_n^{'}}\right)_{\substack{1\le i\le k\\1\le j\le n}},$$

where $\Delta_{F_n^{'}}$ is the right ideal of $U(F_n)$ generated by $F_n^{'}$. If we delete $n-$th column of the matrix $\bar{J}(W)$ we obtain the $k\times(n-1)$ matrix

$$\bar{J}(\overline{W}) = \left(\frac{\partial a_i}{\partial x_j} + \Delta_{F_n^{'}}\right)_{\substack{1\le i\le k\\1\le j\le n-1}}.$$

Let $\frac{\partial a_i}{\partial x_j} \equiv u_{ij}\left(mod\ \Delta_{F_n^{'}}\right)$. Then we have $\frac{\partial a_i}{\partial x_j} - u_{ij} \in \Delta_{F_n^{'}}$. Since the elements a_i do not involve x_n, for each $i, 1\le i\le k$ we get

$$\frac{\partial a_i}{\partial x_j} - u_{ij} \in \Delta_{F_{n-1}^{'}}.$$

Therefore the matrix $\bar{J}(\overline{W})$ can be considered over $U(F_{n-1}/F_{n-1}^{''})$. Put $U = \left(u_{ij}\right)$. Passing to $U(F_{n-1}/F_{n-1}^{''})$ we obtain

$$\bar{J}(W) \equiv U\ \left(mod\ \Delta_{F_{n-1}^{'}}\right).$$

Hence

$$\bar{J}(W).\bar{P} \equiv U.\bar{P}\ \left(mod\ \Delta_{F_{n-1}^{'}}\right).$$

Since the entries of the matrices U and $\bar{P}$ depend on only to $x_1, x_2, \cdots, x_{n-1}$, we have

$$\bar{J}(W).\bar{P} = I_k$$

over $U(F_{n-1}/F_{n-1}^{''})$. Thus $\bar{J}(W)$ is right invertible over $U(F_{n-1}/F_{n-1}^{''})$. Then from Theorem 2.3, W is primitive in M_{n-1}.

We can extend this technique to free solvable Lie algebras.

Corollary 3.3. Let $L_{n-1} = F_{n-1}/\delta^c(F_{n-1}),\ L_n = F_n/\delta^c(F_n)$ and

$$W = \{a_1 + \delta^c(F_{n-1}),\ a_2 + \delta^c(F_{n-1}), \cdots, a_k + \delta^c(F_{n-1})\} \subset L_{n-1}.$$

If the system $\overline{W} = \{a_1 + \delta^c(F_n),\ a_2 + \delta^c(F_n), \cdots, a_k + \delta^c(F_n)\}$ is primitive in L_n then W is primitive in L_{n-1}.

Proof. Let the system $\overline{W}$ be primitive in L_n. By Theorem 2.3 $J(\overline{W})$ is right invertible over $U(F_n/\delta^{c-1}(F_n))$. Using similar arguments as in the proof of Theorem 3.2 we see that $J(\overline{W})$ is right invertible over $U(F_{n-1}/\delta^{c-1}(F_n-1))$. This completes the proof.

3.3. The Frre Nilpotent Case: $R = \gamma_c(F_n)$

It is well known that an endomorphism of a free nilpotent Lie algebra is an automorphism if and only if the linear part of the endomorphism is invertible.

Let $S_n = F_n/\gamma_c(F_n)$ and $S_{n-1} = F-1_n/\gamma_c(F_{n-1}),\ c\ge 2.$

Theorem 3.4. Let

$$W = \{a_1 + \gamma_c(F_{n-1}),\ a_2 + \gamma_c(F_{n-1}), \cdots, a_k + \gamma_c(F_{n-1})\} \subset S_{n-1}.$$

If the system $\overline{W} = \{a_1 + \gamma_c(F_n),\ a_2 + \gamma_c(F_n), \cdots, a_k + \gamma_c(F_n)\}$ is primitive in S_n then W is primitive in S_{n-1}.

Proof. Since the system $\overline{W}$ is primitive in S_n then it is included by a free generating set of S_n. That is, there is a subset $\{b_{k+1} + \gamma_c(F_n),\ b_{k+2} + \gamma_c(F_n), \cdots, b_n + \gamma_c(F_n)\}$ of S_n so that

$$Y = \{a_1 + \gamma_c(F_n),\ a_2 + \gamma_c(F_n), \cdots, a_k + \gamma_c(F_n),\ b_{k+1} + \gamma_c(F_n),\ b_{k+2} + \gamma_c(F_n), \cdots, b_n + \gamma_c(F_n)\}$$

Is a free generating set of S_n. Let

$$a_i = \sum_{j=1}^{n-1} \alpha_{ij} x_j + u_j,\ \text{ where } u_j \in F_n^{'}/\gamma_c(F_n^{'}),\ \alpha_{ij} \in K,\ 1\le i\le k$$

and

$$b_i = \sum_{j=1}^{n} \beta_{ij} x_j + v_j,\ \text{ where } v_j \in F_n^{'}/\gamma_c(F_n^{'}),\ \beta \in K,\ k+1\le i\le n.$$

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Since Y is a free generating set, the matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(n-1)} & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2(n-1)} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta_{(k+1)1} & \beta_{(k+1)2} & \cdots & \beta_{(k+1)(n-1)} & \beta_{(k+1)n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{n(n-1)} & \beta_{nn} \end{pmatrix}$$

Is invertible over the field K. Therefore there exists a marix B such that $A.B = I_n$. Thus the rows of A are linearly independent. This allows us to choice a free generating set of S_{n-1} containing the set $a_1 + \gamma_c(F_{n-1}),\ a_{12} + \gamma_c(F_{n-1}), \cdots, a_k + \gamma_c(F_{n-1})$ as a subset. Hence

$$\{a_1 + \gamma_c(F_{n-1}),\ a_{12} + \gamma_c(F_{n-1}),\ \cdots, a_k + \gamma_c(F_{n-1})\ \}$$

is primitive in S_{n-1}.

Along the same lines as in the proof of Theorem 3.4 one can prove the following.

Corollary 3.5. Let $S_m = F_m/\gamma_c(F_m)$, $S_{m+r} = F_{m+r}/\gamma_c(F_{m+r})$ and

$$W = \{a_1 + \gamma_c(F_m),\ a_2 + \gamma_c(F_m), \cdots, a_k + \gamma_c(F_m)\} \subset S_m.$$

If $\overline{W} = \{a_1 + \gamma_c(F_{m+r}),\ a_2 + \gamma_c(F_{m+r}), \cdots, a_k + \gamma_c(F_{m+r})\}$ is primitive in S_{m+r} then W is primitive in S_m.

Corollary 3.6. Let R be a verbal ideal of F_n, $\hat{R} = R \cap F_{n-1}$ and

$$W = \{a_1 + \widehat{R},\ a_2 + \widehat{R}, \cdots, a_k + \widehat{R}\} \subset F_{n-1}/\widehat{R}.$$

If the system W is primitive in $F_n/\widehat{R}$ then W is primitive in$F_{n-1}/\widehat{R}$.

We conclude this work by summarizing all results which we have already obtained:

Theorem 3.7. . Let R be a verbal ideal of F_n, $\hat{R} = R \cap F_{n-1}$, $W = \{a_1 + \hat{R}, a_2 + \hat{R}, \cdots, a_k + \hat{R}\} \subset F_{n-1}/\hat{R}$ and let $\overline{W} = \{a_1 + R, a_2 + R, \cdots, a_k + R\}$ be primitive in F_n/R. Then W is primitive in $F_{n-1}/\hat{R}$ if

a) $R = \{0\}$

b) F_n/R is free metabelian.

c) F_n/R is free solvable.

d) F_n/R is free nilpotent.

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